ELEN E3401: Electromagnetics

Spring 2025

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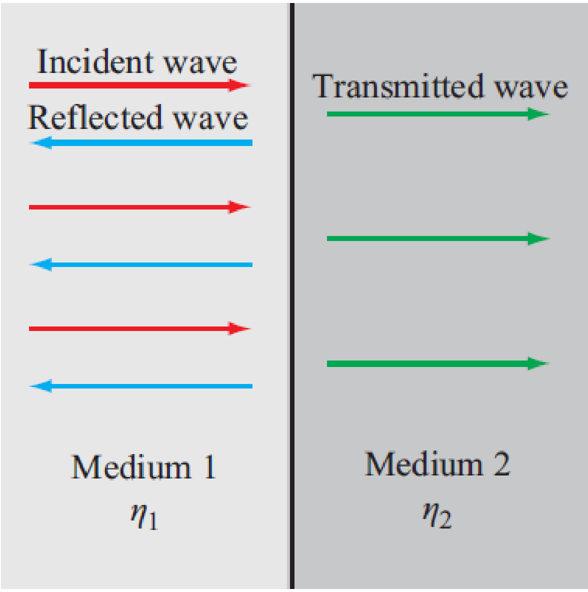
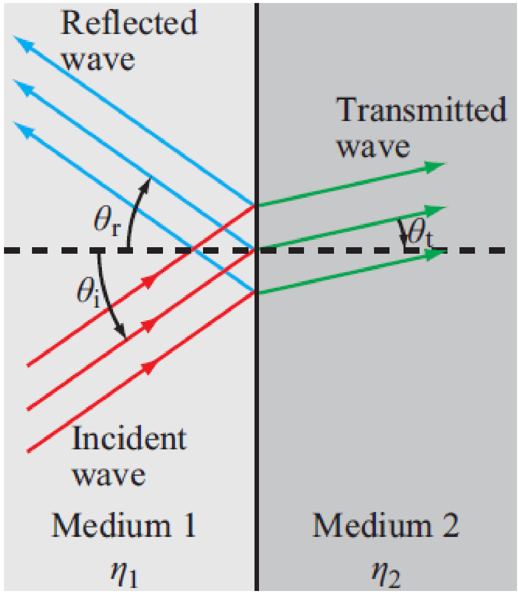
Lecture #22

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

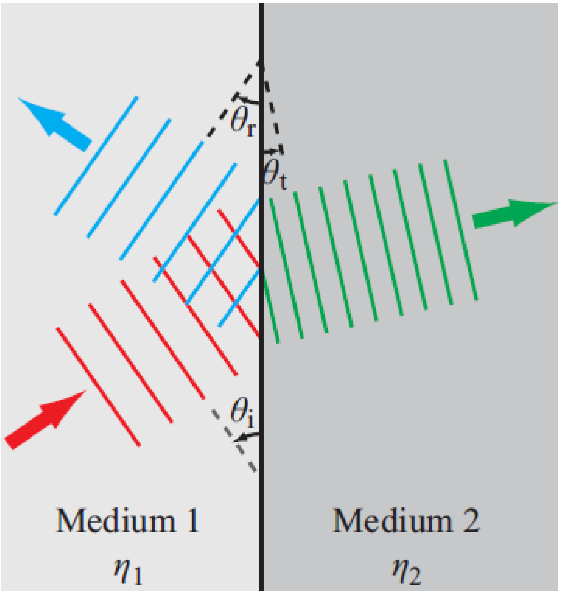
Wave reflection and transmission at interfaces



(a) Normal incidence

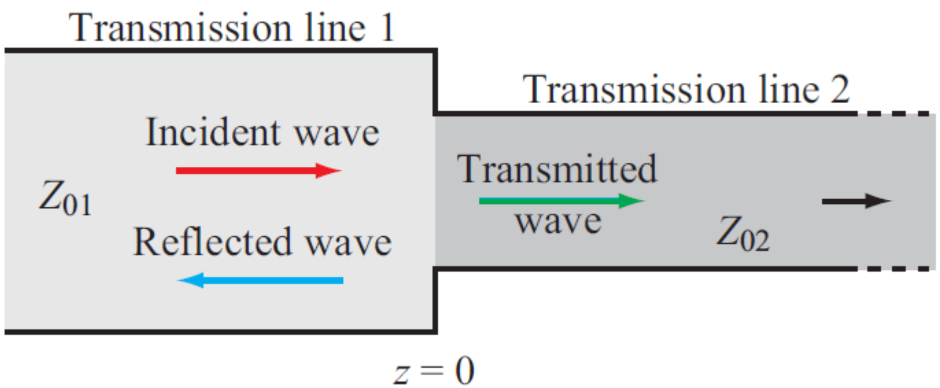


(b) Ray representation of oblique incidence

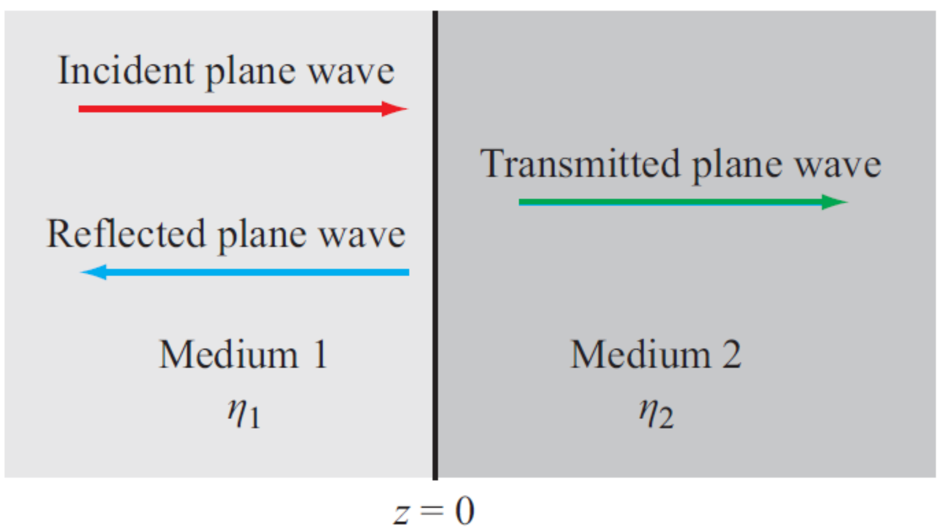


(c) Wavefront representation of oblique incidence

Normal incidence – analogy with transmission line

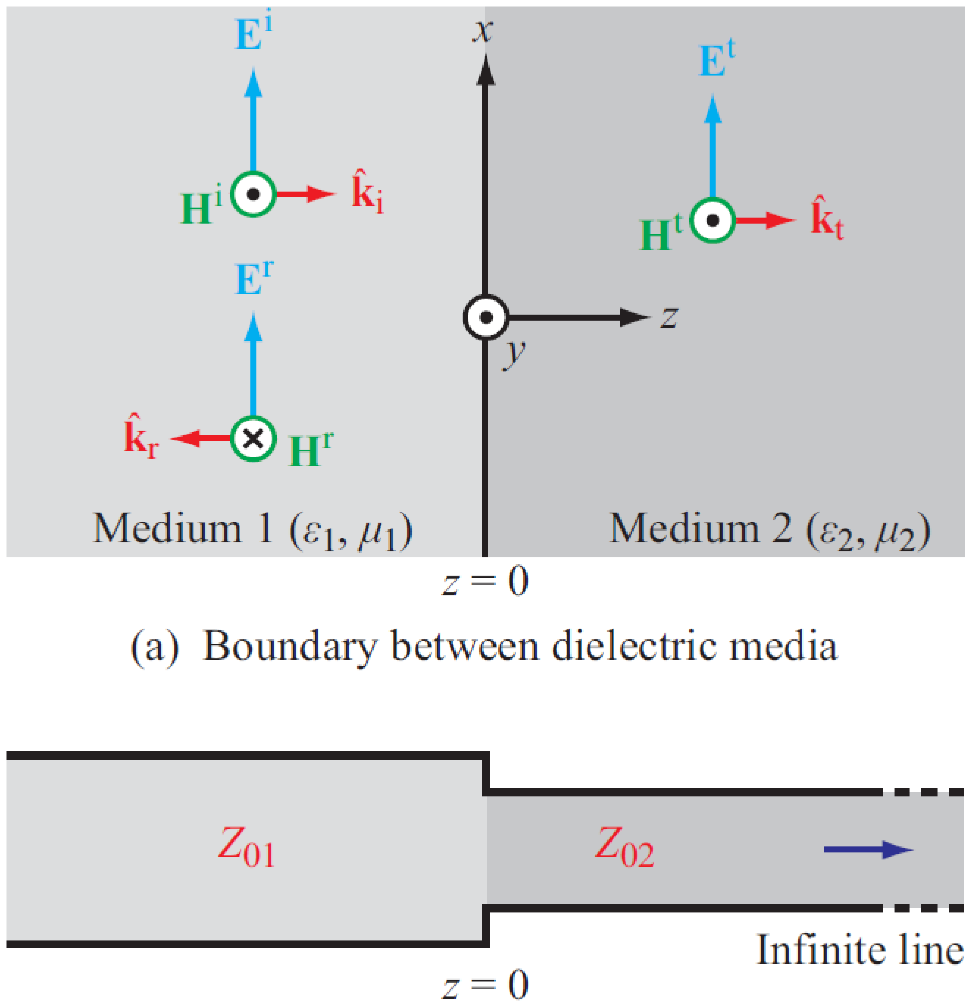


(a) Boundary between transmission lines



(b) Boundary between different media

Normal incidence wave reflection/transmission



(a) Boundary between dielectric media

(b) Transmission-line analogue

Wave reflection/transmission

2 media are lossless, homogeneous, dielectrics

Incident wave: $\tilde{E}^i(z) = \hat{x}E_0^i e^{-jk_1 z}$ $\quad \tilde{H}^i(z) = \hat{z} \times \dfrac{\tilde{E}^i}{\eta_1} = \dfrac{\hat{y}E_0^i}{\eta_1} e^{-jk_1 z}$

Reflected wave: $\tilde{E}^r(z) = \hat{x}E_0^r e^{jk_1 z}$ $\quad \tilde{H}^r(z) = (-\hat{z}) \times \dfrac{\tilde{E}^r}{\eta_1} = \dfrac{-\hat{y}E_0^r}{\eta_1} e^{jk_1 z}$

Transmitted wave: $\tilde{E}^t(z) = \hat{x}E_0^t e^{-jk_2 z}$ $\quad \tilde{H}^t(z) = \hat{z} \times \dfrac{\tilde{E}^t(z)}{\eta_2} = \dfrac{\hat{y}E_0^t}{\eta_2} e^{-jk_2 z}$

$$E_0^i, E_0^r, E_0^t \rightarrow \text{amplitudes}$$

$$k_1 = \omega\sqrt{\mu_1\epsilon_1} \qquad k_2 = \omega\sqrt{\mu_2\epsilon_2}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \qquad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$$

Apply boundary conditions at z = 0

Medium 1 Incident + Reflected

$$\tilde{E}_1(z) = \tilde{E}^i(z) + \tilde{E}^r(z)$$

$$\tilde{E}_1(z) = \hat{x}(E_0^i e^{-jk_1 z} + E_0^r e^{jk_1 z})$$

$$\tilde{H}_1(z) = \tilde{H}^i(z) + \tilde{H}^r(z)$$

$$\tilde{H}_1(z) = \hat{y}\frac{1}{\eta_1}(E_0^i e^{-jk_1 z} - E_0^r e^{jk_1 z})$$

Medium 2 Transmitted

$$\tilde{E}_2(z) = \tilde{E}^t(z) = \hat{x}E_0^t e^{-jk_2 z}$$

$$\tilde{H}_2(z) = \tilde{H}^t(z) = \frac{\hat{y}E_0^t}{\eta_2} e^{-jk_2 z}$$

Boundary conditions at $z = 0$: Tangential components of E, H continuous

$$\tilde{E}_1(0) = \tilde{E}_2(0) \rightarrow E_0^i + E_0^r = E_0^t$$

$$\tilde{H}_1(0) = \tilde{H}_2(0) \rightarrow \frac{E_0^i}{\eta_1} - \frac{E_0^r}{\eta_1} = \frac{E_0^t}{\eta_2}$$

$$E_0^r = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_0^i = \Gamma E_0^i \qquad E_0^t = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right) E_0^i = \tau E_0^i$$

Normal Incidence

$$\Gamma = \frac{E_0^r}{E_0^i} = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)$$ Reflection coefficient

$$\tau = \frac{E_0^t}{E_0^i} = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right)$$ Transmission coefficient

Similar form as for transmission lines

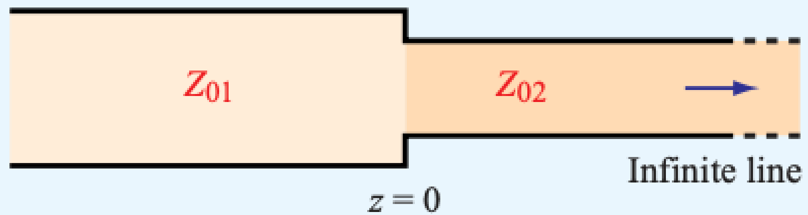
Γ, τ may be <u>complex</u> for conductive media

$$\tau = 1 + \Gamma \quad \text{(normal incidence)}$$

For non-magnetic media, $\eta_1 = \dfrac{\eta_0}{\sqrt{\epsilon_{r1}}}$ $\eta_2 = \dfrac{\eta_0}{\sqrt{\epsilon_{r2}}}$ Free space impedance

$$\Gamma = \frac{\sqrt{\varepsilon_{r_1}} - \sqrt{\varepsilon_{r_2}}}{\sqrt{\varepsilon_{r_1}} + \sqrt{\varepsilon_{r_2}}} \quad \text{(nonmagnetic media).}$$

Transmission Line Analogue



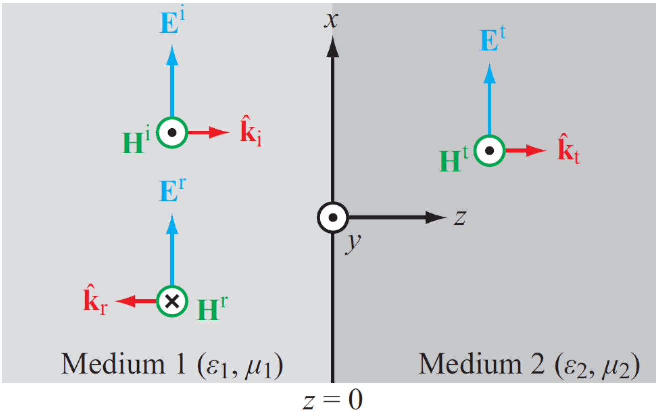
(b) Transmission-line analogue

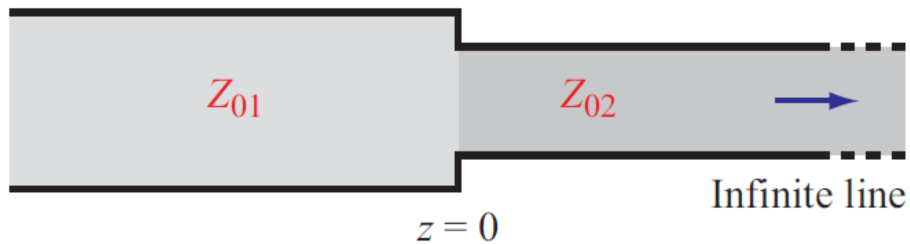
From TL: $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

For infinite line lossless TL input impedance is the characteristic impedance:

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad \tau = 1 + \Gamma$$

Transmission Line Analogue – normal incidence





$$\beta_1 = \omega\sqrt{\mu_1\epsilon_1} \qquad \beta_2 = \omega\sqrt{\mu_2\epsilon_2}$$

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \qquad \tau = 1 + \Gamma$$

ω

$$\widetilde{\boldsymbol{E}}_{\boldsymbol{1}}(z) = \hat{x}E^i_0(e^{-jk_1z} + \Gamma e^{jk_1z})$$

$$\widetilde{V}_1(z) = V_0{}^+(e^{-j\beta_1z} + \Gamma e^{j\beta_1z})$$

$$\widetilde{\boldsymbol{H}}_{\boldsymbol{1}}(z) = \hat{y}\frac{E^i_0}{\eta_1}(e^{-jk_1z} - \Gamma \quad e^{jk_1z})$$

$$\tilde{I}_1(z) = \frac{V_0{}^+}{Z_{01}}(e^{-j\beta_1z} - \Gamma e^{j\beta_1z})$$

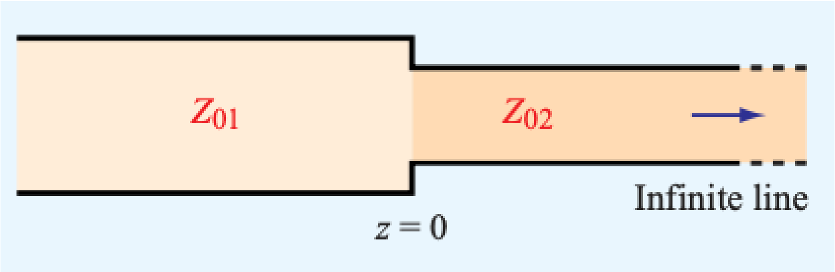
$$\widetilde{\boldsymbol{E}}_{\boldsymbol{2}}(z) = \hat{x}\tau E^i_0 e^{-jk_2z}$$

$$\widetilde{V}_2(z) = \tau V_0{}^+(e^{-j\beta_2z})$$

$$\widetilde{\boldsymbol{H}}_{\boldsymbol{2}}(z) = \hat{y}\tau \quad \frac{E^i_0}{\eta_2}e^{-jk_2z}$$

$$\tilde{I}_2(z) = \frac{\tau V_0{}^+}{Z_{02}}(e^{-j\beta_2z})$$

Transmission Line Analogue



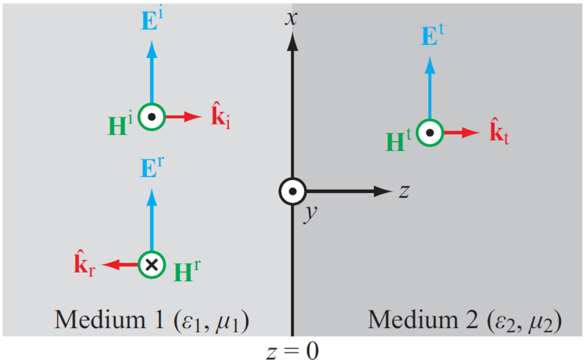
(b) Transmission-line analogue

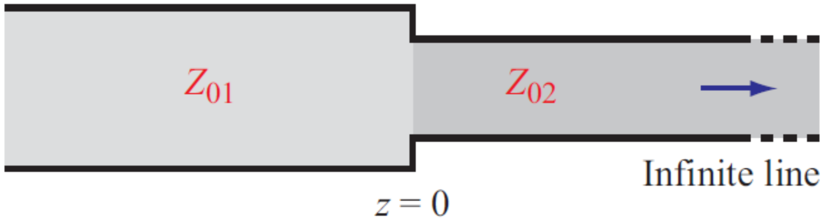
Standing wave ratio: incident + reflected waves $$S = \frac{|\tilde{E}_1|_{max}}{|\tilde{E}_1|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

If 2 media have equal $\eta_1 = \eta_2$ $\Gamma = 0, S = 1$

If medium #2 is perfect conductor: $\eta_2 = 0$ $\Gamma = -1, S = \infty$

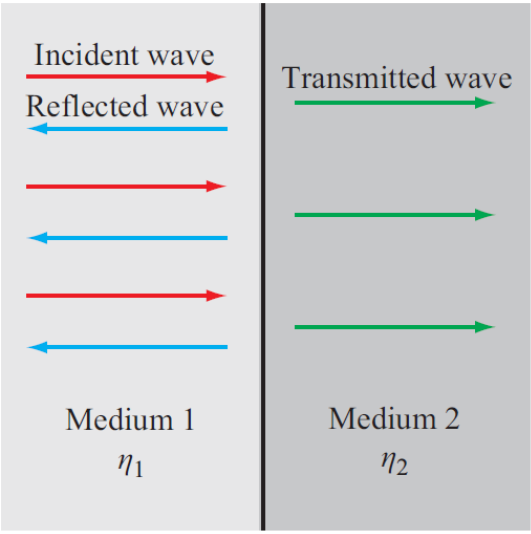
Transmission Line Analogue - summary





Plane Wave [Fig. 8-4(a)]	Transmission Line [Fig. 8-4(b)]
$\widetilde{\mathbf{E}}_1(z) = \hat{\mathbf{x}} E_0^i (e^{-jk_1 z} + \Gamma e^{jk_1 z})$ (8.5a)	$\widetilde{V}_1(z) = V_0^+ (e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z})$ (8.5b)
$\widetilde{\mathbf{H}}_1(z) = \hat{\mathbf{y}} \dfrac{E_0^i}{\eta_1} (e^{-jk_1 z} - \Gamma e^{jk_1 z})$ (8.6a)	$\widetilde{I}_1(z) = \dfrac{V_0^+}{Z_{01}} (e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z})$ (8.6b)
$\widetilde{\mathbf{E}}_2(z) = \hat{\mathbf{x}} \tau E_0^i e^{-jk_2 z}$ (8.7a)	$\widetilde{V}_2(z) = \tau V_0^+ e^{-j\beta_2 z}$ (8.7b)
$\widetilde{\mathbf{H}}_2(z) = \hat{\mathbf{y}} \tau \dfrac{E_0^i}{\eta_2} e^{-jk_2 z}$ (8.8a)	$\widetilde{I}_2(z) = \tau \dfrac{V_0^+}{Z_{02}} e^{-j\beta_2 z}$ (8.8b)
$\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1)$	$\Gamma = (Z_{02} - Z_{01})/(Z_{02} + Z_{01})$
$\tau = 1 + \Gamma$	$\tau = 1 + \Gamma$
$k_1 = \omega\sqrt{\mu_1 \varepsilon_1}$, $k_2 = \omega\sqrt{\mu_2 \varepsilon_2}$	$\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1}$, $\beta_2 = \omega\sqrt{\mu_2 \varepsilon_2}$
$\eta_1 = \sqrt{\mu_1/\varepsilon_1}$, $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$	Z_{01} and Z_{02} depend on transmission-line parameters

Power Flow



Medium 1: $\vec{S}_{av_1}(z) = \frac{1}{2} Re\left[\tilde{E}_1(z) \times \widetilde{H}_1^*(z)\right]$

$$\vec{S}_{av_1}(z) = \frac{1}{2} Re\left[\hat{x} E^i_0 (e^{-jk_1 z} + \Gamma e^{jk_1 z}) \times \hat{y} \frac{{E^i_0}^*}{\eta_1} (e^{jk_1 z} - \Gamma^* e^{-jk_1 z})\right]$$

$$\vec{S}_{av_1}(z) = \hat{z} \frac{|E^i_0|^2}{2\eta_1} (1 - |\Gamma|^2)$$

$$\vec{S}_{av_1} = \vec{S}^i_{av} + \vec{S}^r_{av} \quad \left\{ \begin{aligned} \vec{S}^i_{av} &= \hat{z} \frac{|E^i_0|^2}{2\eta_1} \\ \vec{S}^r_{av} &= -\hat{z}\, |\Gamma|^2 \frac{|E^i_0|^2}{2\eta_1} \end{aligned} \right\} \quad \vec{S}^r_{av} = -\,|\Gamma|^2 \vec{S}^i_{av}$$

Power Flow

Medium 2:

$\tau = 1 + \Gamma$ $\Gamma \rightarrow$ real for lossless, can be complex for conducting

$$\vec{S}_{av_2}(z) = \frac{1}{2} Re\left[\hat{x}\tau E^{i}_{0} e^{-jk_2 z} \times \hat{y}\tau^* \frac{E^{i*}_{0}}{\eta_2} e^{jk_2 z}\right] = \hat{z}|\tau|^2 \frac{|E^{i}_{0}|^2}{2\eta_2}$$

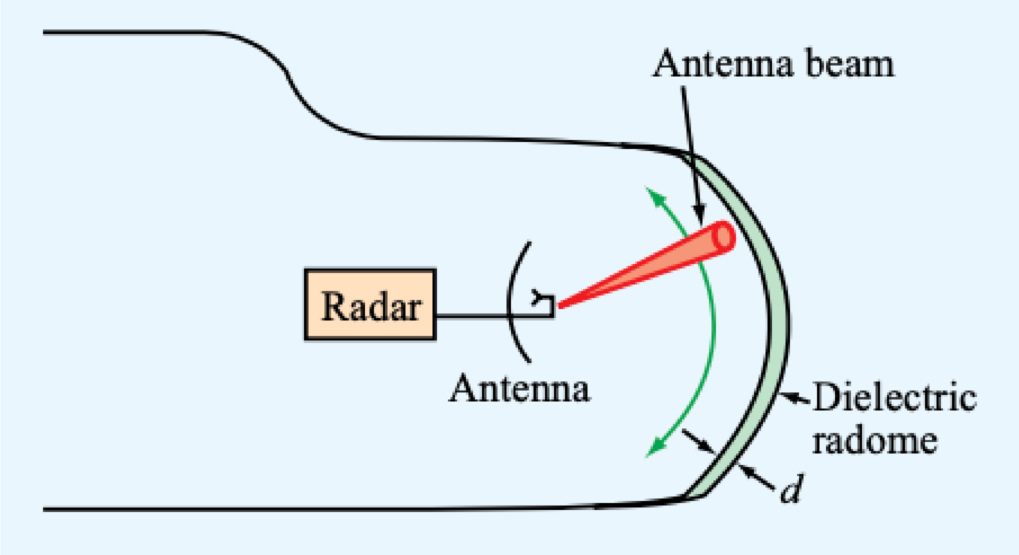
$$\vec{S}_{av_1}(z) = \hat{z}\frac{|E^{i}_{0}|^2}{2\eta_1}(1 - |\Gamma|^2)$$

$$\boxed{\frac{\tau^2}{\eta_2} = \frac{1 - \Gamma^2}{\eta_1} \qquad \text{(lossless media),}}$$

leads to

$$\mathbf{S}_{av_1} = \mathbf{S}_{av_2}.$$

Radar Radome Design

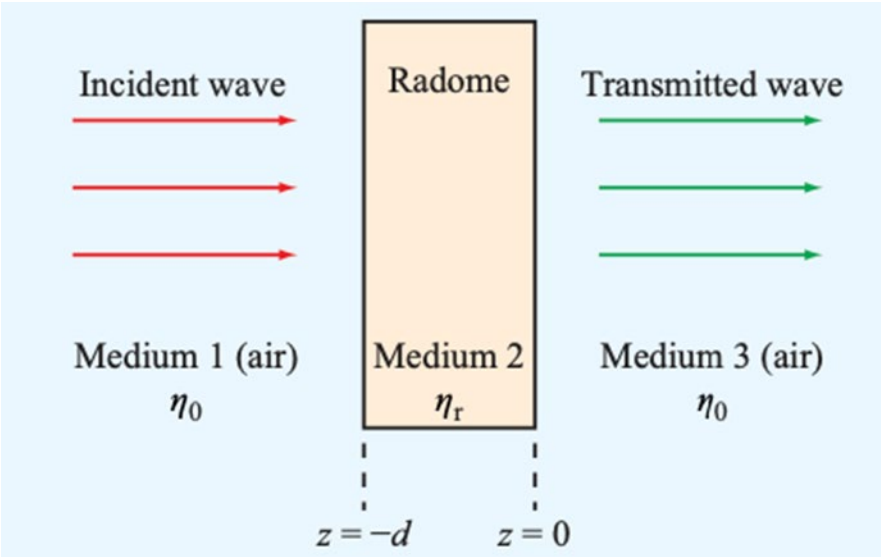


Aircraft uses 10 GHz radar – narrow beam scanning (or gimbal glass) – behind dielectric radome

Assume radome is planar

Design thickness of radome so that it is <u>completely transparent</u> to radar

Radar Radome Design



Medium 1 = air, η_0

Medium 2 = radome, η_r (thickness d)

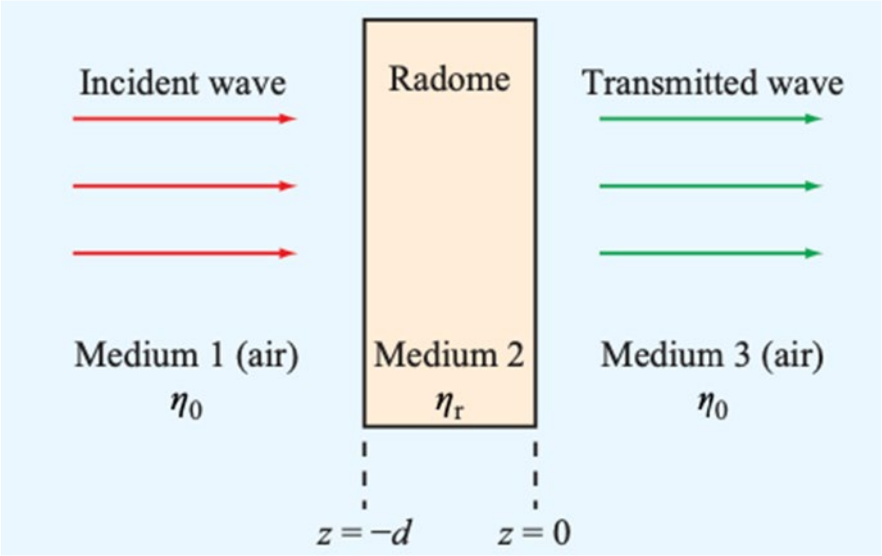
Medium 3 = air, η_0

z=0 → at back surface of radome

$\epsilon_r = 9, \mu_r = 1$

needs to be $d > 2.3$cm to be structurally sound

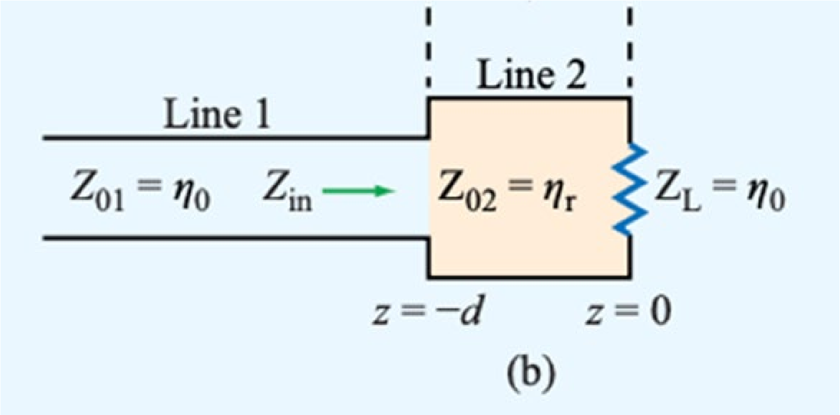
Radar Radome Design

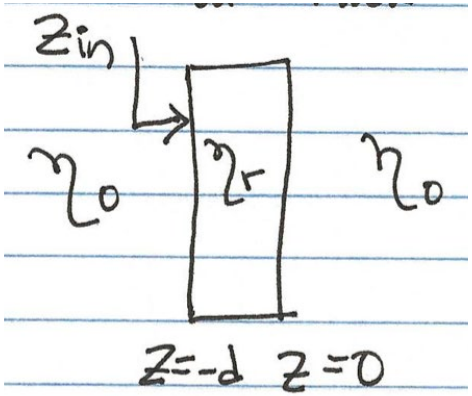


$$\epsilon_r = 9, \mu_r = 1$$

z=0 → at back surface of radome

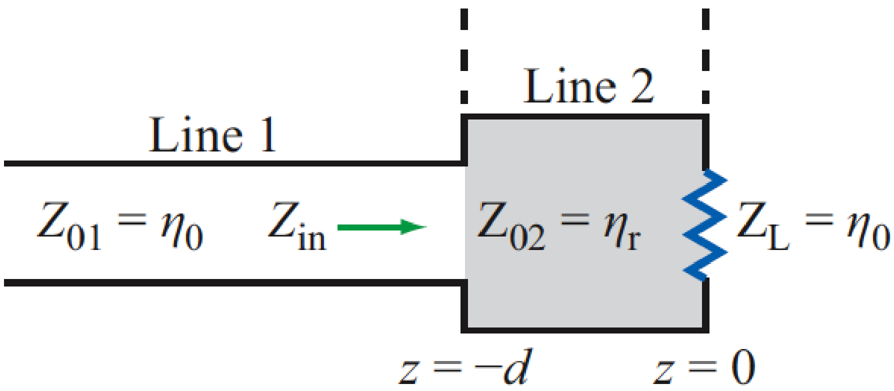
Equivalent TL model





Radar Radome Design

To achieve "transparent" to radar, need $\Gamma = 0$ at $z = -d$



$Z_L = \eta_0$ Why?

Because medium 3 is air

How do we get $\Gamma = 0$?

if $Z_{in}(at\ z = -d) = \eta_0$

How do we get $Z_{in} = \eta_0$

When radome thickness is an integer multiple of $\frac{\lambda}{2}$

Radar Radome Design

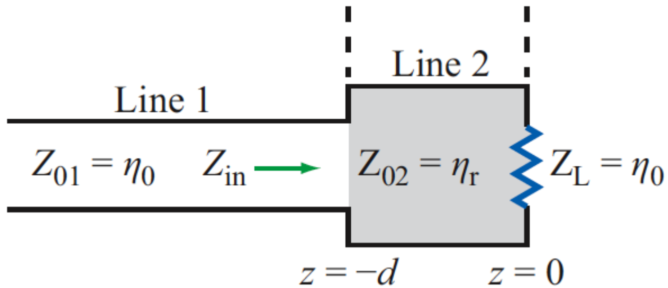
From transmission lines, since Media 1 and 3 are the same (air), no net reflection will occur at $z = -d$ if the radome thickness is an integer multiple of $\frac{\lambda}{2}$

Need half-wave $\frac{\lambda}{2}$ transformer: Z_{in} *of* line, length $l = n\frac{\lambda}{2}$, $(n = 0,1,2, \dots)$

$$Z_{in} = Z_L$$

$$d = l(\text{halfwave}) = n\frac{\lambda_2}{2}$$

Wavelength in medium 2



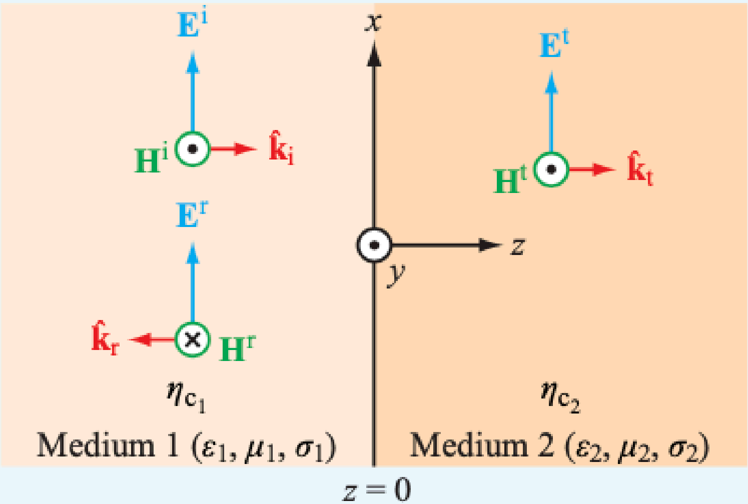
At 10 GHz, $\lambda_0 = \frac{c}{f} = 3cm$ $\quad \lambda_2 = \frac{\lambda_0}{\sqrt{\epsilon_r}} = \frac{3cm}{3} = 1cm$

We choose: $d = 5\frac{\lambda_2}{2} = 2.5cm$

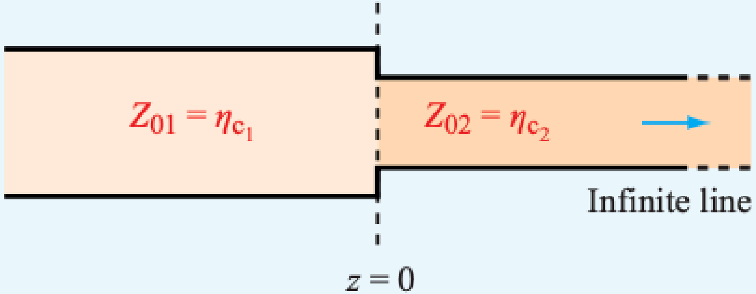
needs to be $d > 2.3$cm to be structurally sound

Radome is non-reflecting, transparent and structurally stable

Boundary between lossy media



(a) Boundary between dielectric media



(b) Transmission-line analogue

Lossy medium: (ϵ, μ, σ)→ constitutive parameters

$$\gamma = \alpha + j\beta$$

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]\right\}^{1/2}$$

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]\right\}^{1/2}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$$

→ With special cases – low loss, good conductors

Boundary between lossy media

<u>Fields in medium 1</u> $jk \rightarrow \gamma, \quad \eta \rightarrow \eta_c$

$$\tilde{E}_1(z) = \hat{x}E^i_0(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}) \qquad \tilde{H}_1(z) = \hat{y}\frac{E^i_0}{\eta_{c_1}}(e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z})$$

<u>Medium 2</u>

$$\tilde{E}_2(z) = \hat{x}\tau E^i_0 e^{-\gamma_2 z} \qquad \tilde{H}_2(z) = \hat{y}\tau\frac{E^i_0}{\eta_{c_2}}e^{-\gamma_2 z}$$

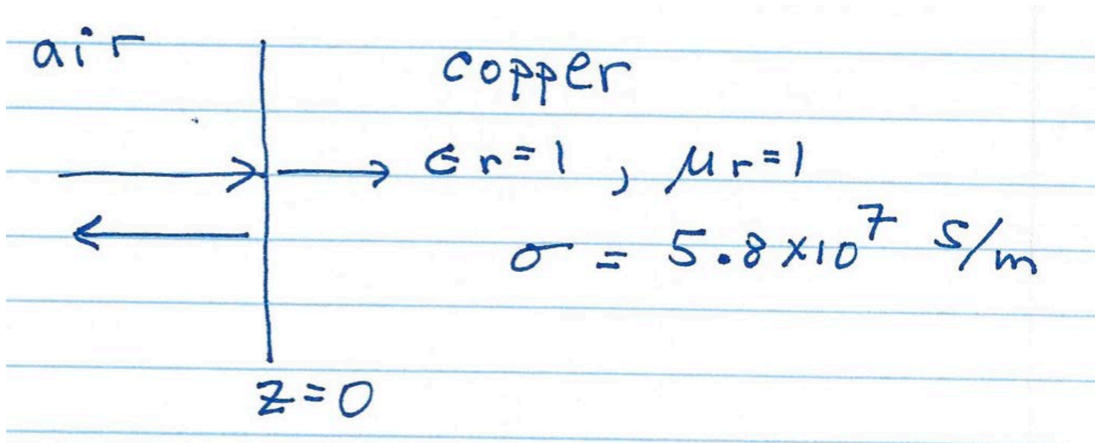
$$\gamma_1 = \alpha_1 + j\beta_1 \qquad \gamma_2 = \alpha_2 + j\beta_2$$

$$\Gamma = \left(\frac{\eta_{c_2} - \eta_{c_1}}{\eta_{c_2} + \eta_{c_1}}\right) \qquad \tau = 1 + \Gamma = \left(\frac{2\eta_{c_2}}{\eta_{c_2} + \eta_{c_1}}\right)$$

$\Gamma, \tau \rightarrow$ complex

Normal incidence on metal

1 GHz, $\hat{x}$ polarized plane wave, travelling in $+\hat{z}$ direction from air to copper surface



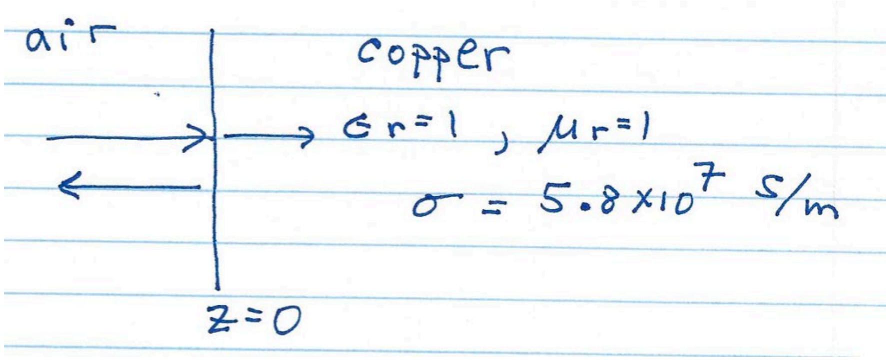
Incident electric field amplitude = 12 mV/m

copper: $\epsilon_r = 1, \mu_r = 1, \sigma = 5.8 \times 10^7 S/m$

Obtain instantaneous $E_1(z,t), H_1(z,t)$ in the air medium.
Assume copper is several δ_s deep

Normal incidence on metal

1 GHz, $\hat{x}$ polarized plane wave, travelling in $+\hat{z}$ direction from air to copper surface



$\epsilon_r = 1, \mu_r = 1, \sigma = 5.8 \times 10^7 S/m$

Electric field amplitude = 12 mV/m

In medium 1 (air): $\alpha = 0$

$$\beta = k_1 = \frac{\omega}{c} = \frac{2\pi \times 10^9}{3 \times 10^8} = \frac{20\pi}{3}\ rad/m$$

$$\eta_1 = \eta_0 = 377\Omega \qquad \lambda = \frac{2\pi}{k_1} = 0.3m$$

Medium 2 (copper): $\sigma = 5.8 \times 10^7 S/m$ $\quad f$=1 GHz $\quad \epsilon_r = 1$

$$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon_r\epsilon_0} = \frac{5.8 \times 10^7}{(2\pi \times 10^9)(10^{-9}/36\pi)} = 1 \times 10^9 \gg 1 \ \rightarrow \text{excellent conductor}$$

Normal incidence on metal

$$\alpha = \sqrt{\pi f \mu \sigma}$$

$$\beta = \alpha = \sqrt{\pi f \mu \sigma}$$

$$\eta_c = (1+j)\frac{\alpha}{\sigma}$$

Good conductor approximation

$$\eta_{c_2} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1+j)\sqrt{\frac{\pi \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} = 8.25(1+j) \quad [m\Omega]$$

This is very small compared to η_{c_1}(air)= 377Ω

→ Copper acts like <u>short circuit</u>

$$\Gamma = \left(\frac{\eta_{c_2} - \eta_0}{\eta_{c_2} + \eta_0}\right) \approx -1$$

Normal incidence on metal

We can approximate $\Gamma = -1$ (like short circuit)

$$\tilde{E}_1(z) = \hat{x}E_0^i(e^{-jk_1z} - e^{jk_1z}) \qquad \tilde{E}_1(z) = -\hat{x}j2E_0^i\sin(k_1\mathrm{z})$$

$$\widetilde{H}_1(z) = \hat{y}\frac{E_0^i}{\eta_1}(e^{-jk_1z} + e^{jk_1z}) \qquad \widetilde{H}_1(z) = \hat{y}\frac{2E_0^i}{\eta_1}\cos(k_1\mathrm{z})$$

With $E_0^i = 12\ \left(\frac{mV}{m}\right)$

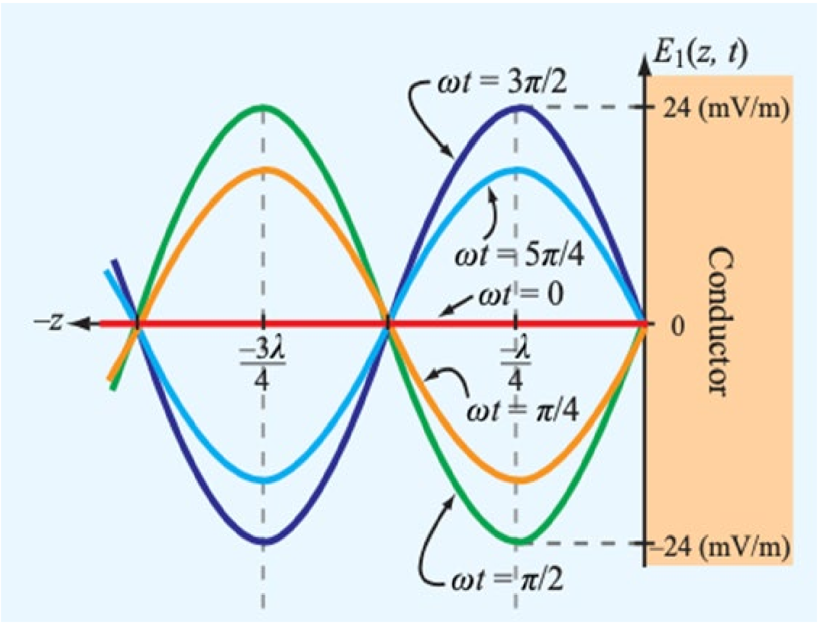
$$\tilde{E}_1(z,t) = Re\left[\tilde{E}_1(z)e^{j\omega t}\right] = \hat{x}2E_0^i\sin(k_1\mathrm{z})\sin(\omega\mathrm{t})$$

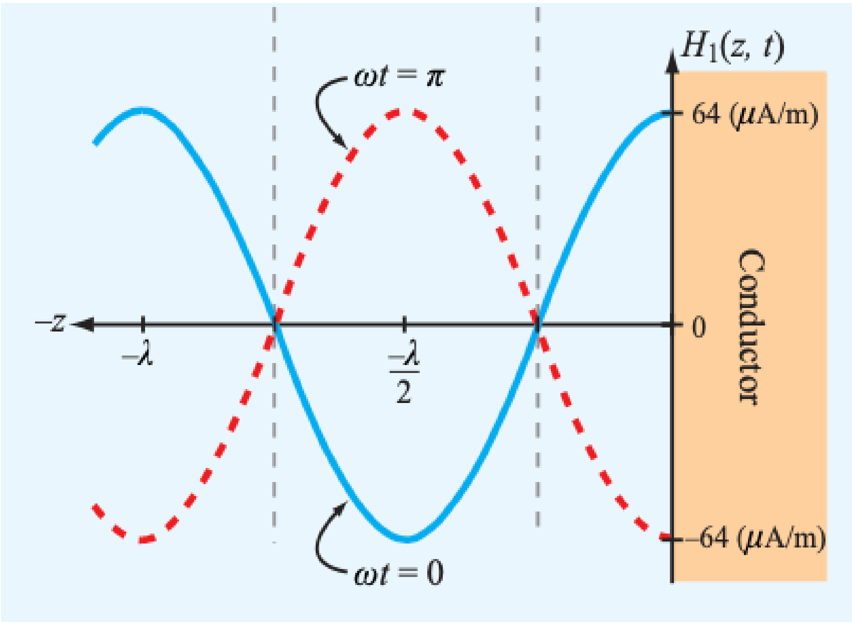
$$\tilde{E}_1(z,t) = \hat{x}24\sin\left(\frac{20\pi}{3}\mathrm{z}\right)\sin(2\pi\times 10^9\mathrm{t})\ \left[\frac{\mathrm{mV}}{\mathrm{m}}\right]$$

$$\widetilde{H}_1(z) = Re\left[\widetilde{H}_1(z)e^{j\omega t}\right] = \hat{y}\frac{2E_0^i}{\eta_{c_1}}\cos(k_1\mathrm{z})\cos(\omega\mathrm{t})$$

$$\widetilde{H}_1(z) = \hat{y}64\cos\left(\frac{20\pi}{3}\mathrm{z}\right)\cos(2\pi\times 10^9\mathrm{t})\left[\frac{\mu A}{\mathrm{m}}\right]$$

Normal incidence on metal





$E_1(z,t)$ *and* $H_1(z,t)$ → **90 degrees phase shift**

Course Project

Wed April 23 – Class will be Projects Workshop

Part 3: Monday May 5: Final Project Presentations (40%)
Each team will have a slot for presentation on their projects, each member must speak for 5min. An additional 3min at the beginning of the presentation should be dedicated to the introduction of the topic and sub-topics. An additional 3min at the end will be used for questions and discussion.

Part 4: DUE Friday May 9 (30%)
Submit paper on the specific task, including quantitative analysis. The papers will be prepared and submitted individually by every student, however team discussion is appropriate. Papers are limited to 6 pages inclusive of figures (references are not counted in the 6 pages).